Brun's Theorem

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Final Presentation

Outline

Twin Primes

Definition (Twin primes)

If p is a prime such that $p + 2$ is also prime, we say p is an twin prime. Let $\mathbb{P}_2$ be the set of twin primes.

Statement of Brun's Theorem

Theorem (Brun's Theorem)

There exists a constant B such that

$$\sum_{p \in \mathbb{P}_2} \frac{1}{p} = B.$$

In other words, the sum of the reciprocals of the twin primes diverges.

Viggo Brun

The mathematician Viggo Brun was born in Sweden in 1885. He is known for his outstanding contributions to the field of number theory.

Brun grew up in rural Sweden and showed a strong interest in mathematics at a young age. He studied mathematics at Stockholm University, where he received his PhD.

Viggo Brun's Early Career

In his early career, Brun focused on analytic number theory and prime number theory. One of his most famous achievements was Brun's theorem in 1915, which gave an upper bound on the distance between prime numbers. This achievement made him an important figure in mathematics at the time.

Brun's Constant

$$B \approx 1.9$$

The value of B is not known exactly and, it is not even known whether it is rational or irrational.

Of course, irrationality would imply that the sum is infinite.

The Pentium Bug

$$B \approx 1.90216058. \qquad \text{(Thomas Nicely (1995))}$$

To be on the safe side, Nicely performed all his calculations twice, using two algorithms on two separate computers, and discovered a bug in the new Intel Pentuium chip.

The Pentium Bug

The Pentium gave incorrect values:

$$\frac{1}{824,633,702,441} \qquad \text{and} \qquad \frac{1}{824,633,702,443}$$

with errors in the tenth decimal place.

Merten's Theorem

We will use the following theorem in our proof:

Theorem (Merten's Theorem)

We have

$$A(x) = \sum_{\substack{p \leq x \\ p \in \mathbb{P}}} \frac{1}{p} = \log \log x + \mathcal{O}(1).$$

This is a stronger version of the statement we proved in class – that the sum of the reciprocals of the primes diverges.

Heuristics

Let E_k be the event that k is prime.

Assume: E_n and E_{n+2} are independent (not true!)

Then

$$\Pr(n \in \mathbb{P}_2) = \Pr(E_n \cap E_{n+2}) \sim \frac{1}{\log^2 n}.$$

Heuristics

By partial summation (Abel summation)

$$\sum_{\substack{p \leq x \\ p \in \mathbb{P}_2}} \frac{1}{p} \approx \sum_{2 \leq n \leq x} \frac{1}{n \log^2(n)}$$

This converges by integral test.

Heuristics

In fact, we need much less: if

$$\Pr(n \in \mathbb{P}_2) = \mathcal{O}\left(\frac{1}{\log^{1.5}(x)}\right),$$

then the sum will converge.

The Sieve of Eratosthenes

The Sieve of Eratosthenes is a process to “sieve” out the prime numbers from 2 to x.

For twin primes, we can do a similar process, but this time we sieve out integers that are

$$0 \text{ or } -2 \pmod{p}.$$

Definition of the Brun Sieve

$$E_p = \left\{ n \in \{1, \ldots, \lfloor x \rfloor\} : n \equiv 0 \text{ or } -2 \pmod{p} \right\}.$$

Definition of the Brun Sieve

Primes $\leq \sqrt{x}$ have negligible contribution to $\pi_2(x)$; so

$$\pi_2(x) \ll \left| \bigcap_{p \leq \sqrt{x}} (E_p)^c \right|$$

Inclusion-Exclusion

$$\begin{aligned}\Pr\left(\bigcap_{p\le\sqrt{x}}(E_p)^c\right) = 1 &- \sum_{p\le\sqrt{x}}\Pr(E_p)\\ &+ \sum_{p_1<p_2}\Pr(E_{p_1}\cap E_{p_2})\\ &- \sum_{p_1<p_2<p_3}\Pr(E_{p_1}\cap E_{p_2}\cap E_{p_3})\\ &+ \cdots .\end{aligned}$$

It turns out we will not want to use $\sqrt{x}$ as our bound, so let's replace it with m.

The Chinese Remainder Theorem

For primes $p_1, \ldots, p_k$, with $p_1 p_2 \cdots p_k \ll n$,

$$\Pr(E_{p_1} \cap E_{p_2} \cap \cdots \cap E_{p_k}) \approx \frac{2^k}{p_1 p_2 \cdots p_k}.$$

The Chinese Remainder Theorem

Idea

Use CRT to approximate (truncated) sum from PIE.

Truncation

For $\sum_{p \le m}$, use CRT when #primes is $\le \dfrac{\log x}{\log m}$

Truncate to $\le c \cdot \dfrac{\log x}{\log m}$ terms, for some $c \ll 1$.

Bounding the Truncated Sum

The truncated sum is

$$
\begin{aligned}
&\prod_{p\le m}\left(1-\frac{2}{p}\right) && \text{(main term)}\\
&+\sum_{p_1<p_2<\cdots<p_{t+1}\le m}\frac{2^{t+1}}{p_1p_2\cdots p_{t+1}}\\
&-\sum_{p_1<p_2<\cdots<p_{t+2}\le m}\frac{2^{t+2}}{p_1p_2\cdots p_{t+2}}+\cdots
\end{aligned}
$$

Bounding the Truncated Sum

To bound the main term, we have

$$\prod_{p\leq m}\left(1-\frac{2}{p}\right)=\mathcal{O}\left(\frac{1}{\log^2 m}\right)$$

using the Taylor expansion of $\log(1+x)$ and the Merten's Theorem.

Bounding the Error Term

Let

$$A(m) = \sum_{p \leq m} \frac{1}{p}.$$

The error term can be bounded by

$$\sum_{k=t+1}^{\infty} \frac{2^k A(m)^k}{k!}.$$

Bounding the Error Term

As long as $t \gg \log\log m$ (say $t > 2\log\log m$) asymptotically, this sum is bounded by the first term, which is approximately

$$\frac{2^t(\log\log m)^t}{t!}.$$

Bounding the Error Term

Bound for the truncated sum:

$$\mathcal{O}\left(\frac{1}{\log^2 m}\right) + \mathcal{O}\left(\frac{2^t(\log\log m)^t}{t!}\right).$$

Finishing the Proof

Stirling's formula gives a bound of the form

$$\frac{1}{\log^2 m} + \left(\frac{2e \log\log m}{t}\right)^t$$

Finishing the Proof

We want:

- the main term should dominate the error term,
- the partial summation should converge.

Finishing the Proof

Set

$$m = x^{\frac{1}{g \log \log x}}$$

for some $g > 0$, to get

$$t = \frac{c \log x}{\log m} = gc \log \log x.$$

Finishing the Proof

New bound: $$\frac{g^2(\log\log x)^2}{\log^2 x} + \left(\frac{2e}{gc}\right)^{gc\log\log x}$$

- want $g \gg \frac{2}{c}$.
- We have

$$\left(\frac{2e}{gc}\right)^{gc\log\log x} = (\log x)^{gc\log(2e/gc)}.$$

For the main term to dominate, we need the exponent to be less than -2.

Finishing the Proof

We find that $c = \frac{1}{3}$ and $g = 27$ works!

Conclusion

We get a bound

$$\pi_2(x) \leq \mathcal{O}\left(\frac{x(\log\log x)^2}{\log^2 x}\right).$$

This is much better than

$$\mathcal{O}\left(\frac{x}{\log^{1.5}(x)}\right)$$

so we're done!

Reference

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