THE CHROMATIC POLYNOMIAL

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Abstract. This expository paper is a general introduction to the theory of chromatic polynomials. Chromatic polynomials are defined, their salient properties are derived, and some practicaI methods for computing them are given. A brief mention is made of the connection between the theory of chromatic polynomials and map coloring problems.

1. Introduction

Color a map with k colors such that no adjacent countries have the same color. How many colors do we need? **Ans:** 4-color theorem says we need at most 4.

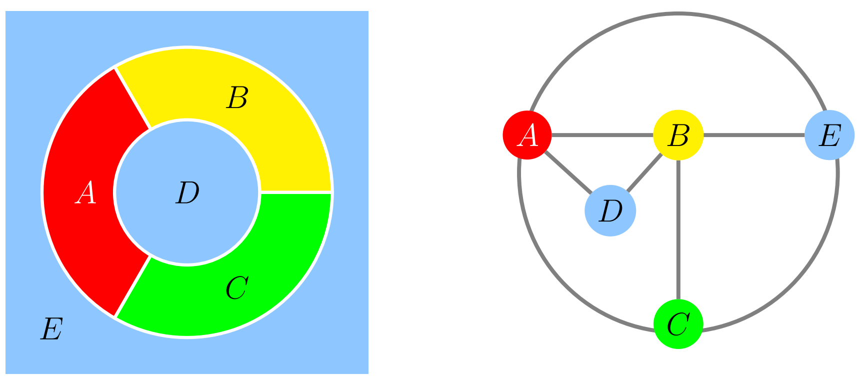

Figure 1. A proper coloring of a map; the corresponding graph labeling.

Graph-theoretic interpretation: in general, how many colors do we need to color a graph? This is a very hard problem, but we can get a lot of interesting theory via the study of chromatic polynomials.

Chromatic polynomials were first defined in 1912 by George David Birkhoff in an attempt to solve the long-standing four colour problem. First, it is necessary to notice that the number of ways a map can be coloured using k colours has a polynomial dependence on k, which we will prove in a rigorous way later in the paper. By viewing maps as loopless planar graphs and defining mathematical properties of colourings on these graphs, Birkhoff hoped to prove that any twodimensional map can be coloured with just four colours so that

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no neighbouring bodies are assigned the same colour. Although he was unsuccessful in this attempt, chromatic polynomials became an important object in algebraic graph theory and continue to be a subject of great interest today.

Birkhoff's definition is limited in that it only defines chromatic polynomials for planar graphs. The concept of chromatic polynomials was later extended in by Hassler Whiteney 1932 to graphs which cannot be embedded into the plane.

Today, the chromatic polynomial has been studied in many novel forms. We are now able to define properties of graphs with more interesting geometries, and on the frontier there has even been study of generalizations of the problems as are seen in hypergraphs or fractional colourings.

2. Basic concepts

Definition. A *proper coloring* of a graph G is a labeling of the graph's vertices with colors such that no two vertices sharing the same edge have the same color. A proper coloring using *at most* k colors is called a (*proper*) k*-coloring*. It is understood that the labels are drawn from the integers $\{1, 2, \ldots, k\}$.

Definition. For $k \in \mathbb{N}$, define $\chi(G; k)$ to be the number of proper k-colorings of G.

Example. $\chi(\overline{K}_n; k) = k^n$ and $\chi(K_3; k) = k(k-1)(k-2)$.

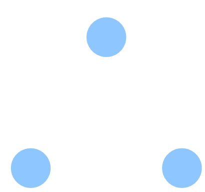

Figure 2. Null graph $\overline{K_3}$

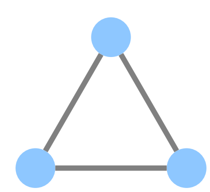

Figure 3. Complete graph K_3

If we want to color the null graph $\overline{K_3}$ with k colors, we notice that this can be done in k^3 ways because there are k color options for each vertex since no vertex is adjacent to another. In general,

$$\chi(\overline{K_n}; k) = k^n.$$

For the complete graph K_3, we begin by selecting a random vertex and note that it can be colored k ways. If we move from this vertex to any other, we notice that this second one can only be colored $k-1$ ways as it is adjacent to the first. The third and final vertex can only be colored $k-2$ ways as it is adjacent to both of the first two (See Figure 3). As a result, we find that K_3 can be colored in $k(k-1)(k-2)$ ways with k or fewer colors. In general,

$$\chi(K_n;k) = k(k-1)(k-2)\cdots(k-n+1).$$

Example. $\chi(P_3;k) = k(k-1)^2 = k^3 - 2k^2 + k$.

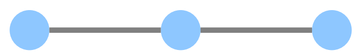

FIGURE 4. P_3: The path graph on 3 vertices

For the path graph P_3, we start with an end vertex and note that this vertex can be colored in k ways. As we move across the graph to the right, each successive vertex can be colored $(k-1)$ ways as it cannot be the same color as the vertex to its left (See Figure 4). Thus, P_3 can be colored $k(k-1)^2$ ways with k or fewer colors. In general,

$$\chi(P_n;k) = k(k-1)^{n-1}.$$

The expression $\chi(G;k)$ is called the **chromatic polynomial** because it turns out to always be a polynomial in k. To prove this, we first prove a very important relation called **Deletion-Contraction** or **chromatic reccurence**.

Recall that for a graph G and edge e, the **deletion** $G-e$ is the subgraph of G with edge e removed. The **contraction** $G \cdot e$ is the graph obtained from G when we contract the endpoints u, v of e into a single vertex.

Note: contractions can cause multiple edges, which don't affect colorings at all, so when we contract a graph, delete all multiple edges.

In order to see how Deletion-Contraction works, consider the following graph G:

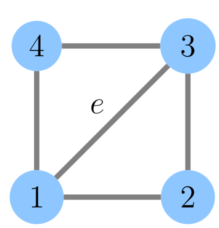

FIGURE 5A. A graph G

Now suppose that the edge e which we want to perform Deletion–Contraction on is the one that connects vertices 1 and 3. Then for the deletion, we simply remove e from G to get the graph $G-e$. For the contraction $G \cdot e$, we once more remove e, but we must also combine

1 and 3 into the same vertex while maintaining all connections that both vertices originally had with 3 and 4. When we perform this contraction, we also remove the multiple edges that would have been created. So the deletion and contraction look like this:

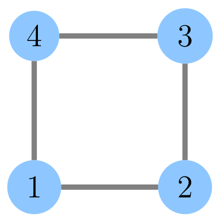

FIGURE 5B. The deletion $G - e$

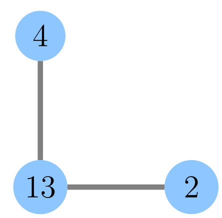

FIGURE 5C. The contraction $G \cdot e$

we know that the chromatic polynomial for G is given by the difference of the chromatic polynomials for $G - e$ and $G \cdot e$. As we have seen $\chi(G - e; k) = (k-1)^4 + (k-1)$ and $P(G \cdot e; k) = k(k-1)^2$. Thus,

$$\chi(G; k) = (k-1)^4 + (k-1) - k(k-1)^2.$$

3. MAIN RESULTS

Theorem 1 (Chromatic Recurrence)**.** *For graph G and edge e of G,*

$$\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k).$$

Chromatic recurrence expresses the chromatic polynomial of a large graph in terms of chromatic polynomials of smaller graphs, so we can use it to compute any chromatic polynomial.

Example. $\chi(P_3; k) = k(k-1)^2 = k^3 - 2k^2 + k$.

This time we will compute $\chi(P_3; k)$ using deletion-contraction.

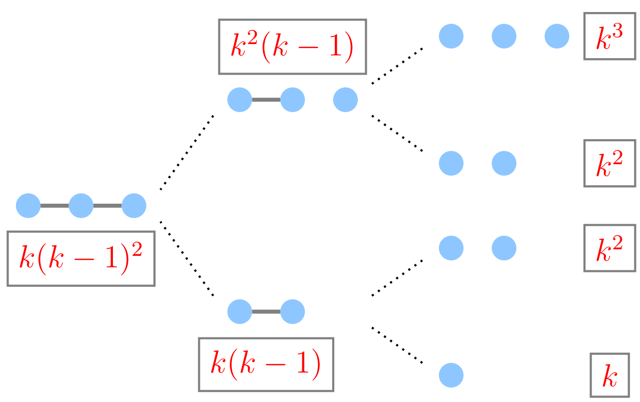

FIGURE 6. Deletion-contraction on P_3.

Using this, we can easily prove $\chi(G;k)$ is a polynomial by induction on $e(G)$.
If we compare the chromatic polynomials of $\overline{K_3}$, P_3, and K_3, we notice that they have some interesting properties.

$$\begin{aligned}\chi(\overline{K_3};k) &= k^3\\ \chi(P_3;k) &= k(k-1)^2 = k^3-2k^2+k\\ \chi(K_3;k) &= k(k-1)(k-2) = k^3-3k^2+2k.\end{aligned}$$

In each of the polynomials above we notice that there is no constant term. Thus, if $k=0$, $\chi(G;0)=0$, as we would expect. Also, except in the case of the null graph, we notice that the sum of the coefficients of each polynomial is 0, which tells us that $\chi(G;1)=0$. This, again, is as expected because any graph with more than 1 vertex and at least one edge cannot be properly colored with only one color. Our final two observations are that the coefficients of these polynomials are integers and have alternating signs and that the absolute value of the coefficient on the term k^{n-1} is the number of edges of the graph. We prove that these characteristics are common to the Chromatic Polynomials of all graphs.
If a polynomial $p(k)$ is negative for positive integer k, then it can't be a chromatic polynomial because there is no such thing as negative number of colorings! Similarly if $p(k) \notin \mathbb{Z}$ for $k \in \mathbb{N}$ then it's not allowed because there's no such thing as a non-integer number of colorings. So chromatic polynomials must satisfy certain properties. Let's look at some neccasary conditions.

Theorem 2. *For a (simple) graph G with $n := n(G)$ vertices and $e(G)$ edges,*

1. *$\chi(G;k)$ has degree n and the leading coefficient of $\chi(G;k)$ (coefficient of x^n) is 1,*
2. *The coefficient of x^{n-1} is $-e(G)$,*
3. *The coefficients are integers and alternate sign,*
4. *The constant term is zero.*
5. *If a coefficent is 0, then all coefficients after it are also zero.*

Proof. These can all be proved by chromatic recurrence and strong induction on the size $e(G)$ of G. □

Example. It is easily seen that these properties hold for $\chi(P_3;k) = k^3-2k^2+k$.

Finally, we prove chromatic recurrence.

Proof of Theorem 1. Consider proper colorings of $G-e$. If the endpoints x,y of e have different colors, then if we add back the edge e, it becomes a proper coloring of G. If x,y have the same color, then contracting them into a single vertex yields a proper coloring of $G\cdot e$. Moreover this process is reversible: for a proper coloring of G or $G\cdot e$ we can assign a unique coloring of $G-e$. It follows

$$\chi(G;k)+\chi(G\cdot e;k)=\chi(G-e;k),$$

which rearranges to chromatic reccurence. □

4. Unsolved Problems

The study of chromatic polynomials of graphs was initiated by Birkhoff in 1912 and continued by Whitney in 1932. Inspired by the four-colour conjecture, Birkhoff and Lewis obtained results concerning the distribution of the real zeros of chromatic polynomials of planar graphs and made the stronger conjecture that they have no real zeros greater than or equal to 4. Their hope was that results from analysis and algebra could be used to prove their stronger conjecture, and hence to deduce that the four-colour conjecture was true. This has not yet occurred: indeed, the four-colour conjecture is now a theorem, but the stronger conjecture of Birkhoff and Lewis remains open. Nevertheless, many beautiful results on chromatic polynomials have been obtained, and many other intriguing questions remain unanswered.

What makes a polynomial chromatic? In other words, given some polynomial, how can we determine whether or not there exists some graph for which it is the chromatic polynomial? The polynomial

$$k^4 - 3k^3 + 3k$$

seems to fit the general form of a chromatic polynomial, yet there is no graph which has this as its chromatic polynomial. So while we have necessary conditions that we can check a given polynomial against, there is still no set of sufficient conditions.

In the examples, we saw that the chromatic polynomial of a path graph is given by

$$\chi(P_n; k) = k(k-1)^{n-1}.$$

We also have the following

Claim 3. *The chromatic polynomial for a tree graph is also $k(k-1)^{n-1}$.*

Proof. We will use induction on n. For the base case, choose $n = 1$. The claim clearly holds as we have k choices of colour for the single vertex. Assume that the claim holds for a graph on n vertices. Select any leaf node and detach it from the tree. By the induction hypothesis, the chromatic polynomial for the resulting tree will be $k(k-1)^{n-2}$. Now, replace the leaf node. We can assign any colour to this node except for that of its neighbour, and thus have $(k-1)$ choices of colour. Then we get

$$\chi(T)n; k) = k(k-1)^{n-2}(k-1) = k(k-1)^{n-1},$$

so the claim holds by induction on n. □

Thus, we see that two unique graphs might share the same chromatic polynomial. What is then a necessary and sufficient condition for two graphs to have the same chromatic polynomial?

Another unsolved problem in a similar vein is that of determining what numbers can be roots of some chromatic polynomial.

A property that is very noticeable when one has calculated a few chromatic polynomials is that the coefficients first increase in absolute magnitude, and then decrease; two successive coefficients may be equal, but it seems that one never finds a coefficient flanked by larger coefficients, and it is natural to conjecture that the coefficients always behave in this way. It is fairly easy to show that the coefficients are bounded in absolute magnitude by the corresponding coefficients in the chromatic polynomial of the complete graph on the same number of nodes (the proof of this will be left as an exercise for the reader); and certainly these upper bounds first increase and then decrease. But whether this is true for all chromatic polynomials is still an open question.

5. Applications

Work on chromatic polynomials has received fresh impetus in recent years from an interaction with mathematical physics. The chromatic polynomial is a specialization of the Potts model partition function, used by mathematical physicists to study phase transitions. A combination of ideas and techniques from graph theory and statistical mechanics has led to significant new results on both polynomials. Other recent progress has been made through the use of techniques from algebraic geometry to solve a long-standing open problem concerning the coefficients of chromatic polynomials.

Lest it seem that chromatic polynomials are not of any particular practical importance, we end with a possible applications.

Allocation of channels to television stations. Assume that there are k possible channels (frequencies) available for use by the n television stations in a certain country. As is well known, stations that are near to each other cannot use the same channel without causing interference. Thus, given any two stations, it may or may not be the case that they can use the same channel. The problem is to allocate a channel to each station in such a way that any two stations which need to have different channels get different channels.

Let us construct a graph G whose nodes represent the stations. We join two nodes by an edge if and only if the corresponding stations cannot use the same channel. Then any allocation of channels is, effectively, a coloring of G in k colors, and if it is proper then the condition about nearby stations being given different channels is satisfied. Thus the problem reduces to that of coloring a graph, and the chromatic polynomial will give the number of ways of allocating the k channels.

The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors with which G can be properly colored; that is, it is the smallest integer λ for which $\chi_G(\lambda) \neq 0$. In the above example it is the minimum number of channels that will suffice to satisfy the allocation conditions.

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