Chapter XV

Height, Krull Dimension, and Localization

This chapter bridges the gap between abstract ring theory—commutative algebra up to irreducibility and factorization in integral domains—and the geometric intuition of algebraic geometry. Our goal is to introduce three central notions—*prime ideals and height*, *Krull dimension*, and *localization*. We will explore how the "size" of a ring (its Krull dimension) corresponds to the dimensions of geometric objects like curves and surfaces, and how localization allows us to zoom in on specific points. We will then weave these concepts together with concrete geometric examples coming from coordinate rings of algebraic varieties of dimensions 0, 1, and 2, as well as some counter-intuitive examples.

1. Height of a Prime Ideal

Recall that prime ideals generalize irreducible elements: an ideal $\mathfrak{p} \subset R$ is prime if $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. In the polynomial ring $k[x]$, over the field k, the prime ideals are (0) and (f) for irreducible f. In $k[x, y]$, primes already come in several geometric flavors:

- (0), corresponding to the whole affine plane;
- primes like (f), defining irreducible curves;
- maximal ideals $(x - a, y - b)$, corresponding to points.

Thus a chain of prime ideals

$$(0) \subset (f) \subset (x - a, y - b)$$

reflects a chain of geometric specializations: plane → curve → point. The exact nature of this correspondence comes from considering the *zero set* of an ideal: the set of points where $(x-a, y-b)$ vanishes is just $\{(a,b)\}$, whereas the zero set of a single irreducible polynomial f defines a curve, and the zero set of the ideal (0) is the whole plane.

Definition 1.1. The *height* of a prime ideal $\mathfrak{p}$ in a ring R is the maximum length h of a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathfrak{p}$$

in R.

Intuitively, height counts how many independent algebraic constraints are imposed in reaching $\mathfrak{p}$. For example:

- In $k[x]$, maximal ideals $(x-a)$ have height 1.
- In $k[x,y]$, maximal ideals $(x-a, y-b)$ have height 2.

We will make precise these correspondences later on.

Proposition 1.1 (Exercise 2.9)**.** *If R is a UFD, then every prime ideal of height 1 in R is principal.*

Proof. Suppose that R is a UFD, and let $\mathfrak{p}$ be a prime ideal of height 1. If $\mathfrak{p}$ contains no irreducible element then we could generate an infinite sequence

$$a_1,\ a_2,\ a_3,\ \ldots \in \mathfrak{p}$$

such that a_{k+1} is a proper divisor of a_k for each $k = 1, 2, 3, \ldots$. This would then give

$$(a_1) \supsetneq (a_2) \supsetneq (a_3) \supsetneq \cdots,$$

which violates the ACCP. Thus $\mathfrak{p}$ contains an irreducible element, say p. Since R is a UFD, p is prime, so (p) is a prime ideal. Since $\mathfrak{p}$ has height one, and $(p) \subseteq \mathfrak{p}$, we must have $(p) = \mathfrak{p}$, so $\mathfrak{p}$ is principal. □

In a UFD, every height-1 prime ideal is principal, reflecting the fact that codimension-1 subvarieties are cut out by a single equation. This is a special case of Krull's Hauptidealsatz, which asserts more generally that any minimal prime over a principal ideal has height at most 1.

Theorem 1.2 (Krull's Hauptidealsatz)**.** *Let R be a Noetherian ring and let $x \in R$. Then every minimal prime ideal $\mathfrak{p}$ over the principal ideal (x) has height at most 1; that is,*

$$\operatorname{ht}(\mathfrak{p}) \leq 1.$$

This implies that every nonzero element in a Noetherian domain is contained in a prime ideal of height 1.

Proposition 1.3 (Exercise 2.10)**.** *If R is a Noetherian ring, and every prime ideal of height* 1 *in R is principal, then R is a UFD.*

Proof. Since R is Noetherian, R satisfies the ACCP. Thus, we just need to show that every irreducible element a is prime. Let a be irreducible; then by Krull's Hauptidealsatz, a is contained in a prime ideal $\mathfrak{p}$ of height 1. By our assumptions, $\mathfrak{p}$ must be principal. Since a is irreducible, we must have $\mathfrak{p} = (a)$, and since $\mathfrak{p}$ is prime, a must be prime. Hence R is a UFD. □

The proposition shows that local structure can determine global properties like unique factorization. To move from the height of a prime to its local properties, we use localization, which captures the geometry only at that specific prime, turning it into the unique maximal ideal.

2. Localization

Localization is the fundamental tool that allows us to algebraically "zoom in" on a prime ideal $\mathfrak{p}$ and its surrounding neighborhood to study its isolated, local properties.

The following exposition is motivated and influenced by [**Con25**].

Let R be a commutative ring. By a *multiplicative set* $S \subset R$, we mean a subset such that

(i) $1 \in S$; and

(ii) $s, t \in S \Rightarrow st \in S$.

The localization $S^{-1}R$ is the ring that is initial in the category of homomorphisms $f : R \to R'$ such that $f(s)$ is a unit for all $s \in S$; we will provide a construction for this object soon.

Note that we allow $0 \in S$ in the definition of a multiplicative set. However, in this case $S^{-1}R$ is the zero ring (see Theorem 2.4 later on). Happily, this is the only case where $S^{-1}R$ is the zero ring, i.e. when $0 \notin S$, $S^{-1}R$ is nontrivial.

The main point of localizing is to understand the behavior of ideals "around" some area, by killing everything else. For example, when $S = R \setminus \mathfrak{p}$, for a prime ideal $\mathfrak{p}$, the localization $R_{\mathfrak{p}} = S^{-1}R$ lets us investigate the behavior of R "at $\mathfrak{p}$". This idea is fundamental to understanding the geometric perspective of height and Krull dimension. This is also the reason for the name "localization": we are using it to study the *local* behavior of the ring.

In the special case where R is a ring of functions on some space X, $Y \subset X$, and S is the subset of functions which do not vanish on Y, then for $a \in R$ and $s \in S$, the quotient a/s represents a well-defined *germ of a function near Y*: it defines a function on the neighborhood of Y where s is nonzero. Such germs can be added and multiplied, and they form a ring whose algebraic structure reflects the behavior of functions on X in an infinitesimal neighborhood of Y. This is precisely the localization.

When R is not an integral domain, R (and hence S) may have zero divisors. However, this does not break the construction of $S^{-1}R$, since we only require that the elements of S are nonvanishing on Y. In particular, the elements of S need not be globally invertible.

If S has zero-divisors, the homomorphism $\ell : R \to S^{-1}R$ will not be injective. Indeed, if $s \in S$ is a zero divisor, then $as = 0$ for some a. Hence, $\ell(a) = a/1 = as/s = 0/s = 0$. Due to this, the construction of $S^{-1}R$ is more subtle than what may seem like very similar constructions, like the field of fractions of an integral domain. The issue of zero-divisors motivates the construction of $S^{-1}R$, which is as follows:

$$S^{-1}R = \{(a, s) \mid a \in R, s \in S\}/\sim,$$

where $(a_1, s_1) \sim (a_2, s_2)$ if and only if there exists $s \in S$ with

$$a_1 s_2 s = a_2 s_1 s \in R.$$

The homomorphism $\ell : R \to S^{-1}R$ will be defined by

$$\ell(a) = (a, 1).$$

We use a/s to denote the equivalence class of $(a, s) = \ell(a)\ell(s)^{-1}$. Of course, we would like to have

$$(a, s) \sim (at, st) \quad \text{for } t \in S.$$

The usual definition of equality of fractions, $a_1/s_1 = a_2/s_2$ if $a_1 s_2 = a_2 s_1$, would be adequate if no element of S were a zero divisor. The additional factor of t in the definition above allows for the possibility that $a_1 s_2 - a_2 s_1 \neq 0$, but $a_1 s_2 t - a_2 s_1 t = 0$ for some $t \in S$.

Now we verify the details above.

Proposition 2.1 (Exercise 4.7)**.** *We have the following:*

- *The relation $\sim$ is an equivalence relation.*
- *The addition and multiplication operations*

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

  are well-defined.
- *$S^{-1}R$ is a commutative ring and the function $a \mapsto \frac{a}{1}$ defines a ring homomorphism $\ell : R \to S^{-1}R$.*
- *$\ell(s)$ is invertible for all $s \in S$.*

Proof. We start by showing that $\sim$ is an equivalence relation. Reflexivity follows from $tsa = tsa$ for all (a, s) and $t \in S$, so $(a, s) \sim (a, s)$. Symmetry is evident from the definition. For transitivity, suppose $(a_1, s_2) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$. Then $ss_2a_1 = ss_1a_2$ and $ts_3a_2 = ts_2a_3$ for $s, t \in S$, so

$$(sts_2)s_3a_1 = sts_1s_3a_2 = sts_1s_2a_3 = (sts_2)s_1a_3,$$

showing that $(a_1, s_2) \sim (a_3, s_3)$. Thus, $\sim$ is an equivalence relation.

To show that addition and multiplication are well-defined, suppose $(a_1, s_2) \sim (a_2, s_2)$. Then there is some $s \in S$ such that $ss_2a_1 = ss_1a_2$. We have

$$\frac{a_1}{s_1} + \frac{b}{t} = \frac{a_1t + bs_1}{s_1t}$$

and

$$\frac{a_2}{s_2} + \frac{b}{t} = \frac{a_2t + bs_2}{s_2t}.$$

Then,

$$ss_2t(a_1t + bs_1) = ss_2ta_1t + ss_2tbs_1 = ss_2t(a_2t + bs_2),$$

and it follows that

$$\frac{a_1t + bs_1}{s_1t} = \frac{a_2t + bs_2}{s_2t}.$$

Similarly, if $(b_1, t_1) \sim (b_2, t_2)$ then

$$\frac{a}{s} + \frac{b_1}{t_1} = \frac{a}{s} + \frac{b_2}{t_2},$$

which implies that that addition is well-defined. For multiplication, suppose that $(a_1, s_1) \sim (a_2, s_2)$; then $ss_2a_1 = ss_1a_2$ for some $s \in S$. Then, $sa_1bs_2t = sa_2bs_1t$, which implies that $\frac{a_1b}{s_1t} = \frac{a_2b}{s_2t}$. Similarly, $\frac{b_1}{t_1} = \frac{b_2}{t_2}$ implies $\frac{a}{s} \cdot \frac{b_1}{t_1} = \frac{a}{s} \cdot \frac{b_2}{t_2}$, so multiplication is well-defined.

Checking that $S^{-1}R$ is a commutative ring is simple; the ring axioms follow in the same way as in the case for the field of fractions for a domain R.

Moreover, the map ℓ is a ring homomorphism since $\ell(1) = 1/1 = 1$, and

$$\begin{aligned}
\ell(a) + \ell(b) &= \tfrac{a}{1} + \tfrac{b}{1} \\
&= \tfrac{a+b}{1} \\
&= \ell(a + b), \\
\ell(a)\ell(b) &= \tfrac{a}{1} \cdot \tfrac{b}{1} \\
&= \tfrac{ab}{1} \\
&= \ell(ab)
\end{aligned}$$

Finally, for all $s \in S$, note that $\ell(s) \cdot \frac{1}{s} = \frac{s}{s} = 1$, so $\ell(s)$ is a unit. □

We can now prove the universal property for $S^{-1}R$. Consider the category of ring homomorphisms $f : R \to R'$ such that $f(s)$ is invertible in R' for every $s \in S$. The morphisms are given by commutative diagrams

$$\begin{array}{ccc} R' & \xrightarrow{\varphi} & R'' \\ {\scriptstyle f}\uparrow & \nearrow_{g} & \\ T & & \end{array}$$

Proposition 2.2 (Exercise 4.7)**.** *In this category, the map $\ell \to S^{-1}R$ is initial.*

Proof. The proof is very similar to that for the field of fractions. Indeed, suppose we have a homomorphism $j : R \to R'$ such that $j(s)$ is invertible in R' for every $s \in S$. Then the map

$$\widehat{j} : S^{-1}R \to R'$$

is in fact forced because we must have

$$\begin{aligned} \widehat{j}(a/s) &= \widehat{j}(a/1)\widehat{j}((s/1)^{-1}) \\ &= (\widehat{j} \circ \ell(a))(\widehat{j} \circ \ell(s))^{-1} \\ &= j(a)j(s)^{-1}. \end{aligned}$$

The function $\widehat{j}$ indeed exists, since $j(s)$ is always invertible; and it is unique since it is forced. The fact that $\widehat{j}$ is a homomorphism follows directly from the fact that j is, so this proves the universal property. □

Having characterized $S^{-1}R$ by its universal property, we now turn to its basic algebraic features. In particular, we verify that localization preserves the absence of zero divisors.

Proposition 2.3 (Exercise 4.7)**.** *Suppose R is an integral domain. Then $S^{-1}R$ is an integral domain.*

Proof. For nonzero $a/s, b/t \in S^{-1}R$, note that $(a/s)(b/t) = (ab/st)$. Suppose to the contrary that $ab/st = 0 = 0/1$. Then there is some $u \in S$ such that $uab = u \cdot 0 \cdot st = 0$. But this means that either u, a, or b is zero. But a/s, b/t are nonzero, so $a, b \neq 0$. So $u = 0$, so $0 \in S$. But then $ua \cdot 1 = us \cdot 0 = 0$, so $a/s = 0/1 = 0$, a contradiction. Hence, $S^{-1}R$ is an integral domain. □

Proposition 2.4 (Exercise 4.7)**.** *The localization $S^{-1}R$ is the zero ring if and only if $0 \in S$.*

Proof. If $S^{-1}R$ is the zero-ring, then $1/1 = 0/1 = 0$, so there is $u \in S$ so that $u(1 \cdot 1 - 0 \cdot 1) = 0$, i.e. $u = 0$. Hence $0 \in S$. Conversely, if $0 \in S$, then for any $a/s \in S^{-1}R$, $0(a \cdot 1 - 0 \cdot s) = 0$, so $a/s = 0/1 = 0$. Hence $S^{-1}R$ is the zero-ring. □

We can similarly define a notion of localization for modules.

Suppose M is a R-module and let S be a multiplicatively closed subset of R as before. Define a relation $\sim$ on the set of pairs (m, s), where $m \in M$ and $s \in S$ by

$$(m, s) \sim (m', s') \iff (\exists t \in S),\ t(s'm - sm') = 0,$$

similarly to above. Proving that $\sim$ is an equivalence relation is identical to the proof for the localization for rings case; see above. Again, we denote by $\frac{m}{s}$ the equivalence class of (a, s), and define the addition operation on these fractions by

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}.$$

We can define a $S^{-1}R$-module structure on $S^{-1}M$ as follows:

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st},$$

for $a/s \in S^{-1}R$ and $m/t \in S^{-1}M$.

Proposition 2.5 (Exercise 4.8)**.** *This defines a $S^{-1}R$-module structure on $S^{-1}M$ that is compatible with the R-module structure on M.*

Proof. To check that this defines a $S^{-1}R$-module structure, note that

$$\begin{aligned}
\frac{a}{s} \cdot \left(\frac{m_1}{t_1} + \frac{m_2}{t_2}\right) &= \frac{a}{s} \cdot \frac{t_2 m_1 + t_1 m_2}{t_1 t_2} \\
&= \frac{a t_2 m_1 + a t_1 m_2}{s t_1 t_2} \\
&= \frac{(s t_2)(a m_1) + (s t_1)(a m_2)}{(s t_1)(s t_2)} \\
&= \frac{a m_1}{s t_1} + \frac{a m_2}{s t_2} \\
&= \frac{a}{s} \cdot \frac{m_1}{t_1} + \frac{a}{s} \cdot \frac{m_2}{t_2},
\end{aligned}$$

$$\begin{aligned}
\left(\frac{a_1}{s_1} + \frac{a_2}{s_2}\right) \cdot \frac{m}{t} &= \left(\frac{a_1 s_2 + a_2 s_1}{s_1 s_2}\right) \cdot \frac{m}{t} \\
&= \frac{a_1 s_2 m + a_2 s_1 m}{s_1 s_2 t} \\
&= \frac{(s_2 t)(a_1 m) + (s_1 t)(a_2 m)}{(s_1 t)(s_2 t)} \\
&= \frac{a_1 m}{s_1 t} + \frac{a_2 m}{s_2 t} \\
&= \frac{a_1}{s_1} \cdot \frac{m}{t} + \frac{a_2}{s_2} \cdot \frac{m}{t},
\end{aligned}$$

$$\begin{aligned}\left(\frac{a_1}{s_2}\cdot\frac{a_2}{s_2}\right)\cdot\frac{m}{t} &= \frac{a_1a_2}{s_1s_2}\cdot\frac{m}{t}\\ &= \frac{a_1a_2m}{s_1s_2t}\\ &= \frac{a_1}{s_1}\cdot\frac{a_2m}{s_2t}\\ &= \frac{a_1}{s_1}\cdot\left(\frac{a_2}{s_2}\cdot\frac{m}{t}\right),\end{aligned}$$

and

$$1\cdot\frac{m}{t} = \frac{1}{1}\cdot\frac{m}{t} = \frac{1\cdot m}{1\cdot t} = \frac{m}{t}.$$

Furthermore, since

$$\frac{r}{1}\cdot\frac{m}{1} = \frac{rm}{1},$$

this is compatible with the R-module structure on M. □

The following result gives a correspondence between ideals of R disjoint from S and ideals of $S^{-1}R$; this resembles the correspondence between ideals of R containing a given ideal I and ideals of R/I given by the third isomorphism theorem.

Proposition 2.6 (Exercise 4.9)**.** *Let S be a multiplicative subset of a commutative ring R. Let $\ell : R \to S^{-1}R$ be the natural homomorphism and J be a proper ideal of $S^{-1}R$. Let I be an ideal of R disjoint from S. Then*

- *$I^e := S^{-1}I$ is a proper ideal of $S^{-1}R$.*
- *$J^c := \ell^{-1}(J)$ is an ideal of R such that $J^c \cap S = \emptyset$.*
- *We have $(J^c)^e = J$ and $(I^e)^c = \{a \in R : (\exists s \in S)\, sa \in I\}$.*

Notation. The superscripts e and c stand for "extension" and "contraction", respectively.

Proof. Suppose that I is an ideal of R such that $I \cap S = \emptyset$. It is evident that $I^e = S^{-1}I$ is an ideal of $S^{-1}R$, since for any $\frac{r}{t} \in S^{-1}R$,

$$\frac{r}{t}\frac{a}{s} = \frac{ra}{st} \in S^{-1}I.$$

Furthermore, note that if $1 \in S^{-1}I$, then $1 = \frac{a}{s}$ for some $a \in I$, so $a = s \in S$, contradicting $I \cap S = \emptyset$. So $S^{-1}I$ is a proper ideal of $S^{-1}R$.

Next suppose $\ell : R \to S^{-1}R$ is the natural homomorphism, and J is a proper ideal of $S^{-1}R$. Let $J^c = \ell^{-1}(J)$. If $r \in R$ and $a \in J^c$, then since $a \in J$, $ra \in J$, so $ra \in J^c$. Thus, J^c is an ideal of R. Furthermore, if $a \in J^c \cap S$, then $1 = a/a \in J$, contradicting that J is a proper ideal of $S^{-1}R$. Hence, $J^c \cap S = \emptyset$.

Now suppose $a/s \in J$. Then $a \in J$, so $a \in J^c$, so $a/s \in S^{-1}J^c = (J^c)^e$. Conversely, if $x \in (J^c)^e$, then $x = a/s$ for some $a \in J^c \subseteq J$, so $x \in J$. This shows that $(J^c)^e = J$.

Finally, to show that $(I^e)^c = \{a \in R : (\exists s \in S)\, sa \in I\}$, note that I^e consists of all fractions a/s, for $a \in I$. Then, $a \in (I^e)^c$ if and only if $a = b/s$ for some $b \in I$, $s \in S$, which is equivalent to saying $sa \in I$. □

Example 2.1 (Exercise 4.9)**.** We need not have $(I^e)^c = I$. Indeed, let

$$S = \{1, x, x^2, \ldots\}, \quad R = \mathbb{C}[x, y], \quad \text{and} \quad I = (xy).$$

Then, $y \in (I^e)^c$ since there exists $s \in S$ such that $sy \in I$: just take $s = x$. But $y \notin I$, so $I \neq (I^e)^c$.

Localization is designed to discard algebraic information coming from elements of S and to retain only what remains visible after those elements are inverted. Since prime ideals are precisely the algebraic loci where elements vanish, it is natural to expect that prime ideals intersecting S should disappear under localization, while those disjoint from S should survive unchanged. The next proposition makes this precise.

Proposition 2.7 (Exercise 4.10)**.** *The assignment $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ gives an inclusion-preserving bijection between the set of prime ideals of R disjoint from S and the set of prime ideals of $S^{-1}R$.*

Proof. Suppose that $\mathfrak{p}$ is a prime ideal of R disjoint from S. We claim that $\mathfrak{p}^e = S^{-1}\mathfrak{p}$ is prime in $S^{-1}R$. To see this, suppose $(a/s)(b/t) \in \mathfrak{p}^e$. Then $(ab/st) = (p/u)$ for some $p \in \mathfrak{p}$ and $u \in S$. Then, $abu = uap \in \mathfrak{p}$, so either u, a, or b is in $\mathfrak{p}$. But $\mathfrak{p}$ is disjoint from S, so $u \notin \mathfrak{p}$. So a or b is in $\mathfrak{p}$, so (a/s) or (b/s) is in $\mathfrak{p}^e$. Thus, $\mathfrak{p}^e$ is prime.

Now, suppose that $\mathfrak{q}$ is a prime ideal of $S^{-1}R$. We claim that $\mathfrak{q}^c$ is a prime ideal of R, that is disjoint from S. To see this, suppose that $ab = \mathfrak{q}$ for some $q \in \mathfrak{q}^c$. Then, $ab = \mathfrak{q}$ in $\mathfrak{q}$, so either a or b is in $\mathfrak{q}$. Without loss of generality, $a \in \mathfrak{q}$. Thus, $a \in \mathfrak{q}^c$. It follows that $\mathfrak{q}^c$ is prime, and it is disjoint from S due to the previous problem.

Thus, we have well-defined inclusion-preserving maps that send prime ideals of R disjoint from S to prime ideals of $S^{-1}R$, and vice-versa. We claim that these maps are inverses. One direction follows since $(\mathfrak{q}^c)^e = \mathfrak{q}$. Now, consider $(\mathfrak{p}^c)^e = \{a \in R : (\exists s \in S)\,(sa \in \mathfrak{p})\}$ for a prime ideal disjoint from S. Then if $sa \in \mathfrak{p}$, either $s \in \mathfrak{p}$ or $a \in \mathfrak{p}$. The first case is impossible, so $a \in \mathfrak{p}$. Thus, $(\mathfrak{p}^c)^e = \mathfrak{p}$. This completes the other direction. Thus, we have an inclusion-preserving bijection between the set of prime ideals of R disjoint from S. □

A particularly important case is localization at the complement of a prime ideal. For a prime ideal $\mathfrak{p}$ of R, if $S = R \setminus \mathfrak{p}$, we let $R_{\mathfrak{p}} = S^{-1}R$ be the *localization of R at $\mathfrak{p}$*. Note that S is multiplicatively closed, since if $ab \in \mathfrak{p}$ then either a or b is in $\mathfrak{p}$. We similarly define $M_{\mathfrak{p}}$ to be $S^{-1}M$, where M is a R-module.

In this situation, all information away from the chosen prime is inverted, and the resulting ring should capture only the algebraic structure "near" that prime. The following corollary formalizes this idea: localization at $\mathfrak{p}$ collapses all geometric directions except those specializing to $\mathfrak{p}$, yielding a local ring, i.e., a ring with only one maximal (prime) ideal, whose prime ideals record exactly the chain of specializations inside $\mathfrak{p}$. In particular, localization allows one to replace global dimension questions by local ones, with

$$\dim R_{\mathfrak{p}} = \mathrm{ht}(\mathfrak{p}).$$

Corollary 2.8 (Exercise 4.11)**.** *The ring $R_{\mathfrak{p}}$ is local, and there is an inclusion-preserving bijection between prime ideals of $R_{\mathfrak{p}}$ and prime ideals of R contained in $\mathfrak{p}$.*

Proof. The map in Theorem 2.7 gives the desired bijection. Furthermore, since it is inclusion-preserving, it follows that maximal ideals of $R_{\mathfrak{p}}$ correspond to prime ideals contained in $\mathfrak{p}$ that are maximal with respect to inclusion. But $\mathfrak{p}$ is the only such ideal, so the only maximal ideal of $R_{\mathfrak{p}}$ is $S^{-1}\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$. □

A property $\mathcal{P}$ of a ring R (or an R-module M) is said to be *local* if R (or M) has $\mathcal{P}$ if and only if $R_{\mathfrak{p}}$ (or $M_{\mathfrak{p}}$) does, for every prime ideal $\mathfrak{p}$ of M.

We give an example of a local property.

Proposition 2.9 (Exercise 4.12)**.** *Let M be a R-module. The following are equivalent:*

(1) $M = 0$*;*

(2) $M_{\mathfrak{p}} = 0$ *for all prime ideals $\mathfrak{p}$ of R;*

(3) $M_{\mathfrak{m}} = 0$ *for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To show (3) $\Rightarrow$ (1), suppose to the contrary that $M \neq 0$ but $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R. Let x be a non-zero element of M. The annihilator $\mathfrak{a} = \mathrm{Ann}(x)$ is a proper ideal, so it is contained in a maximal ideal $\mathfrak{m}$. Consider the element $x/1 \in M_{\mathfrak{m}}$. Since $M_{\mathfrak{m}} = 0$, it follows that $x/1 = 0$, which means that $bx = 0$ for some $b \in M \setminus \mathfrak{m}$, which is impossible as $b \notin \mathfrak{a}$. This establishes (3) $\Rightarrow$ (1) and completes the proof. □

Furthermore, if $\varphi : M \to N$ is a homomorphism, then injectivity and surjectivity of φ are local properties (see Proposition 3.9 in [**AM69**]).

Now, we investigate how unique factorization behaves under localization.

Proposition 2.10 (Exercise 4.15). *Let R be a UFD and S be a multiplicatively closed subset of R. Then $S^{-1}R$ is a UFD.*

Proof. Let P be the set of all irreducibles in R dividing some element of S, and Q be the set of all irreducibles in R not dividing some element of S. We claim that every element of P is a unit in $S^{-1}R$, and every element of Q is irreducible in $S^{-1}R$. Indeed, if $p \in P$, then $pa = s \in S$ for some a, so $p \cdot (a/s) = 1$ in $S^{-1}R$, implying that p is a unit. Now suppose $q \in Q$. If $q \cdot (a/s) = 1$ for some $a \in R$, $s \in S$, then $qa = s$, so q divides an element of S, which is impossible. Hence q is not a unit. Furthermore, if $(a/s) \cdot (b/t) = q$ in $S^{-1}R$, then $qst = ab$, so $q \mid ab$ in R. Since R is a UFD, all irreducibles are prime, so $q \mid a$ or $q \mid b$. Consequently, either $q \mid a/s$ or $q \mid b/t$ in $S^{-1}R$. It follows that q is prime, hence irreducible, in $S^{-1}R$.

Now, suppose that a/s is irreducible in $S^{-1}R$. Consider the prime factorization of a in R, which is unique up to associates in R. If there are no primes from Q, then a/s is a product of units in $S^{-1}R$, so it is a unit, a contradiction. Otherwise, $q \mid a$ for some $q \in Q$. Then, $q \mid a/s$ in $S^{-1}R$. Since q and a/s are both irreducible, it follows that it is an associate of q.

Now, we prove that $S^{-1}R$ is a UFD. Pick some arbitrary a/s; let

$$a = u(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})(q_1^{e_1} \cdots q_\ell^{e_\ell})$$

be the prime factorization of a in R, where u is a unit in R, $p_i \in P$ and $q_i \in Q$. Then

$$\frac{a}{s} = \frac{u}{s}(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})(q_1^{e_1} \cdots q_\ell^{e_\ell}).$$

From above, $\frac{u}{s}(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k})$ is a unit in $S^{-1}R$, and each term in the product $q_1^{e_1} \cdots q_\ell^{e_\ell}$ is irreducible. Hence, this is gives an irreducible factorization of $\frac{a}{s}$. Now, suppose we have two irreducible factorizations

$$\frac{a}{s} = \frac{a_1}{s_1} \cdots \frac{a_k}{s_k} = \frac{b_1}{t_1} \cdots \frac{b_\ell}{t_\ell}.$$

Then, each a_i is the associate of some $q_i \in Q$, and each b_I is the associate of some $r_i \in Q$. Hence, $q_1 \cdots q_k$ and $r_1 \cdots r_\ell$ are associates in $S^{-1}R$. Since no unit is divisible by an irreducible element, it follows from above that all units have the form $\frac{a}{s}$, where a is a product of primes in P. In particular, it follows that

$$r_1 \cdots r_\ell = \frac{a}{s}(q_1 \cdots q_k),$$

so

$$sr_1 \cdots r_\ell = aq_1 \cdots q_k.$$

Since R is a UFD, the prime factorizations of the left-hand side and right-hand side must match, up to associates. However, s and a are not divisible by any primes in P, so it follows that $q_1, \ldots, q_k$ and $r_1 \cdots r_\ell$ can be rearranged so that q_i and r_i are associates. It follows that the two factorizations of $\frac{a}{s}$ match up to associates. Hence, unique factorization holds. $\square$

Proposition 2.11. *Suppose R is a Noetherian integral domain, and $s \in R$ be a nonzero prime. Consider the multiplicatively closed subset $S = \{1, s, s^2, \dots\}$. Then R is a UFD if and only if $S^{-1}R$ is a UFD.*

Proof. One direction follows from Theorem 2.10. Now, suppose $S^{-1}R$ is a UFD. Since R is Noetherian, by Theorem 1.3, it suffices to prove that every prime ideal of height 1 in R is principal. Let $\mathfrak{p}$ be such a prime ideal. If $s^k \in \mathfrak{p}$, for some k, then since s^k is a product of k copies of s, $s \in \mathfrak{p}$. Hence, $(s) \subseteq \mathfrak{p}$. Since s is prime, (s) is prime, and since $\mathfrak{p}$ has height one, $\mathfrak{p} = (s)$, so $\mathfrak{p}$ is principal. Now suppose $\mathfrak{p}$ is disjoint from S. Then, by Theorem 2.7, $\mathfrak{p}^e$ is a prime ideal of height one in $S^{-1}R$, and since $S^{-1}R$ is a UFD, by Theorem 1.1, $\mathfrak{p}^e = (a)$ is principal. Write $a = b/s$, with $b \in I$. Then $\mathfrak{p}^e = (b)$ as well. For $x \in (\mathfrak{p}^e)^c$, note that $x \in \mathfrak{p}^e$, so $x = sb^k$ for some $s \in S$. Since $\mathfrak{p}$ is prime, $s \in \mathfrak{p}$ or $b^k \in \mathfrak{p}$. But the first case is impossible, so $b^k \in \mathfrak{p}$. Hence, $x \in (b)$. Thus, $\mathfrak{p} = (b)$ is principal. It follows that R is a UFD, and hence R is a UFD if and only if $S^{-1}R$ is. $\square$

Unfortunately, being a UFD is *not* a local property. There exist rings R such that $R_{\mathfrak{p}}$ is a UFD for all prime ideals $\mathfrak{p}$, but R is not a UFD. Such rings are called *locally factorial*.

3. Krull Dimension

Definition 3.1. The *Krull dimension* (or simply dimension) of a ring R is the supremum of the lengths of chains of prime ideals in R. Equivalently, it is the supremum of the heights of prime ideals of R.

We denote the Krull dimension of R by $\dim R$. Some technicalities: we do not count the whole ring as a prime ideal, and we say that a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ has length r. Krull dimension represents the "topological dimension" of $\mathrm{Spec}(R)$.

For an ideal I of R, we define the *dimension* $\dim I$ to be the dimension of R/I. Additionally, by Theorem 2.8, $\mathrm{ht}\,\mathfrak{p} = \dim R_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$.

From the geometric point of view, the Krull dimension of the coordinate ring of an affine variety coincides with the geometric dimension of that variety—at least in the reduced, irreducible cases.

3.1. The Geometric Perspective.

We now make precise the geometric perspective on Krull dimension. This was mentioned briefly in Example III.4.14 in [**Alu09**], and now we will make everything fully rigorous.

We start by reviewing the correspondence between ideals and zero sets, which is discussed in more detail in §VII.2.3 of [**Alu09**]. Throughout this discussion, let k be an algebraically closed field.

Given an ideal $I \subset k[x_1, \ldots, x_n]$, define its *zero set*

$$V(I) = \{a \in k^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

Such sets are called *algebraic sets*.

Conversely, given a subset $X \subset k^n$, define its *vanishing ideal*

$$I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in X\}.$$

The assignments $I \mapsto V(I)$ and $X \mapsto I(X)$ reverse inclusions: if $I \subset J$, then $V(J) \subset V(I)$. Thus larger ideals correspond to smaller geometric sets.

The fundamental link between algebra and geometry is provided by Hilbert's Nullstellensatz.

Theorem 3.1 (Hilbert's Nullstellensatz, informal form)**.** *For any ideal $I \subset k[x_1, \ldots, x_n]$,*

$$I(V(I)) = \sqrt{I}.$$

Here the radical ideal $\sqrt{I}$ is defined by

$$\sqrt{I} = \{x \in k[x_1, \ldots, x_n] : x^m \in I \text{ for some } m \geq 1\}.$$

In particular, radical ideals correspond bijectively to algebraic subsets of k^n. Moreover:

- prime ideals correspond to *irreducible* algebraic sets;
- maximal ideals correspond to *points*.

This dictionary allows us to interpret chains of prime ideals as chains of irreducible geometric specializations.

The correspondence between ideals and zero sets naturally equips k^n with a topology.

Definition 3.2. The *Zariski topology* on k^n is defined by declaring the closed sets to be the algebraic sets $V(I)$ for ideals $I \subset k[x_1, \ldots, x_n]$.

This topology is very coarse:

- every nonempty open set is dense;
- points are closed, but finite sets need not be open;
- irreducible closed sets play the role of connected components.

Irreducibility has a precise algebraic meaning.

Proposition 3.2. *An algebraic set $X = V(I)$ is irreducible if and only if I is a prime ideal.*

This is not too hard to check using the basic properties of the ideal-variety correspondence.

Thus prime ideals are the basic building blocks of algebraic geometry. Chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$$

correspond to chains of irreducible closed subsets

$$X_r \subsetneq X_{r-1} \subsetneq \cdots \subsetneq X_0,$$

and Krull dimension measures the maximum possible length of such chains. From this perspective, Krull dimension is *intrinsic* to the Zariski topology: it counts how many times one can pass to a strictly smaller irreducible closed subset.

3.2. Affine Varieties and Coordinate Rings. Let k be an algebraically closed field. An *affine algebraic variety* is a subset

$$X \subset k^n$$

of the form

$$X = V(I) = \{a \in k^n \mid f(a) = 0 \text{ for all } f \in I\},$$

for some ideal $I \subset k[x_1, \ldots, x_n]$.

Definition 3.3. The *coordinate ring* of an affine variety X is

$$k[X] := k[x_1, \ldots, x_n]/I(X),$$

where

$$I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in X\}$$

is the ideal of all polynomials vanishing on X.

By Hilbert's Nullstellensatz, $I(X)$ is a radical ideal, and every finitely generated reduced k-algebra arises (up to isomorphism) as the coordinate ring of an affine variety.

The algebraic structure of $k[X]$ encodes the geometry of X:

- maximal ideals of $k[X]$ correspond to points of X;
- prime ideals correspond to irreducible subvarieties of X;
- inclusions of prime ideals correspond to specializations of subvarieties.

Thus chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$$

correspond to chains of irreducible subvarieties

$$X_r \subsetneq X_{r-1} \subsetneq \cdots \subsetneq X_0.$$

Definition 3.4. The *dimension* of an affine variety X is defined to be

$$\dim X := \dim k[X].$$

This definition recovers familiar geometric cases:

- finite sets of points have coordinate rings of dimension 0;
- irreducible affine curves have coordinate rings of dimension 1;
- irreducible affine surfaces have coordinate rings of dimension 2.

In the context of geometry, localizing at a prime ideal $\mathfrak{p}$ (denoted $R_{\mathfrak{p}}$) is akin to using a microscope to look only at the behavior of functions near the point $\mathfrak{p}$. Geometrically, $R_{\mathfrak{p}}$ corresponds to looking at the variety defined by R in an infinitesimal neighborhood of the subvariety defined by $\mathfrak{p}$.

3.3. Examples of Krull Dimension. One of the most basic results about Krull dimension is the following.

Theorem 3.3. *We have*

$$\dim k[x_1, \ldots, x_r] = r.$$

More generally, if R is Noetherian, $\dim R[x] = 1 + \dim R$.

Intuitively, this makes sense: the chain of ideals

$$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_r).$$

In fact, we can say more.

An *affine domain* over k is the coordinate ring $R = k[x_1, \cdots, x_n]/I(X)$ of an *irreducible* affine algebraic variety X. Equivalently, it is the quotient of $k[x_1, \ldots, x_n]$ by a prime ideal.

For a field extension $k \subseteq K$, the *transcendence degree* $\operatorname{trdeg}_k(K)$ of K is the size of a maximal algebraically independent subset of K (over k). It turns out that all such subsets have the same size, so this notion is well-defined. Intuitively, the transcendence degree measures how many variables we need add to obtain K from k. For a k-algebra R, the transcendence degree $\operatorname{trdeg}_k(R)$ is the transcendence degree of the field of fractions of R.

When $R = k[x_1, \ldots, x_r]$, the transcendence degree is easily seen to be r: $\{x_1, \ldots, x_r\}$ is a maximal algebraically independent set in the field of fractions of R (which is just rational functions in $x_1, \ldots, x_r$). So Theorem 3.3 says that the Krull dimension matches the transcendence degree. This can be generalized.

Theorem 3.4. *If R is an affine domain, then* $\dim R = \operatorname{trdeg}_k(R)$. *Furthermore, this is the common length of all maximal chains of prime ideals in R.*

This is a very powerful result which lays down the connection between the algebraic and geometric notions of dimension. A proof of this result is given in [**Eis95**].

The second part of this theorem has a very nice consequence. Suppose $\mathfrak{p}$ is a prime ideal of R. There is a maximal chain of prime ideals of $R/\mathfrak{p}$ of length $\dim \mathfrak{p}$. Prime ideals of $R/\mathfrak{p}$ correspond to prime ideals of R that contain $\mathfrak{p}$, so we may convert this into a chain of prime ideals starting from $\mathfrak{p}$. Furthermore, by adjoining this with a maximal chain of prime ideals ending at $\mathfrak{p}$, we get a maximal chain of prime ideals, which has length $\operatorname{ht} \mathfrak{p} + \dim \mathfrak{p}$. But since all maximal chains of prime ideals have length $\dim R$, it follows that

$$\dim R = \operatorname{ht} \mathfrak{p} + \dim \mathfrak{p}. \tag{3.1}$$

Hence, in this case the height equals the *codimension* $\dim R - \dim \mathfrak{p}$.

Example 3.1 (Dimension 0). A ring has Krull dimension 0 if and only if all its prime ideals are maximal. Such rings are precisely the Artinian rings.

Let $R = k[x]/(f(x))$, where f is squarefree and splits over k. Writing $f(x) = (x - a_1) \cdots (x - a_n)$, note that

$$R = \frac{k[x]}{(x - a_1)(x - a_2) \cdots (x - a_n)}.$$

It is straightforward to check that the maximal ideals $(x - a_1), \ldots, (x - a_n)$ satisfy the conditions of the Chinese Remainder Theorem (Theorem V.6.1 in [**Alu09**]), from which it follows that

$$R \cong \frac{k[x]}{(x - a_1)} \times \cdots \times \frac{k[x]}{(x - a_n)} \cong k \times \cdots \times k$$

Additionally, $\operatorname{Spec}(R)$ consists of finitely many points: the ideals $(x - a_i)$ for $i = 1, \ldots, n$. Every prime is maximal, and $\dim R = 0$ (which also follows from the fact that $k \times \cdots \times k$ has transcendence degree 0).

This matches geometric intuition perfectly, since a finite collection of points should have dimension zero.

Example 3.2 (Dimension 1). Consider $R = k[x]$. We have

$$(0) \subset (x - a)$$

as the longest possible chain, so $\dim R = 1$. This corresponds to the affine line, as noted in Example III.4.14 in [**Alu09**].

More interestingly, let

$$R = k[x, y]/(y^2 - x^3).$$

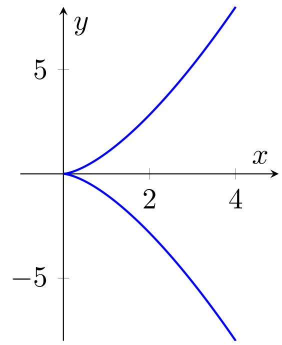

Figure 1. The variety $V(y^2 - x^3)$ has a cusp

This is the coordinate ring of a cusp, as seen in Figure 1.

The ring is an integral domain, and every maximal ideal corresponds to a point on the curve. Here, there is only one independent parameter, so the transcendence degree is 1. Hence, $\dim R = 1$, even though the curve has a singularity.

Example 3.3 (Dimension 2)**.** The ring $k[x, y]$ has dimension 2, reflecting the two independent parameters x and y. More generally, if

$$R = k[x, y, z]/(f)$$

with f irreducible, then R is the coordinate ring of an irreducible surface in $\mathbb{A}^3$, and $\dim R = 2$.

Chains of primes correspond to surface → curve → point; this formalizes the intuition from the beginning of the chapter.

Example 3.4 (Embedded components; **[Eis95]**, Chapter 9, p.228)**.** Consider the ring

$$R = k[x, y, z]/(xy, xz).$$

This is NOT an affine domain, since (xy, xz) is not a prime ideal, so we cannot use Theorem 3.4 to compute the dimension. Instead, we note that the dimension of R is the maximum of the dimensions of the quotients of R by the minimal ideals, which follows from the definitions. The minimal prime ideals in this ring are (x) and (y, z), and since $R/(x) \cong k[y, z]$ has dimension 2, and $R/(y, z) \cong k[x]$ has dimension 1, it follows that $\dim R = 2$.

We can also see this geometrically. Note that R is the coordinate ring for the space defined by $xy = 0$ and $xz = 0$. It is not hard to see R as the union of the plane

$$M = \{(x, y, z) : x = 0\}$$

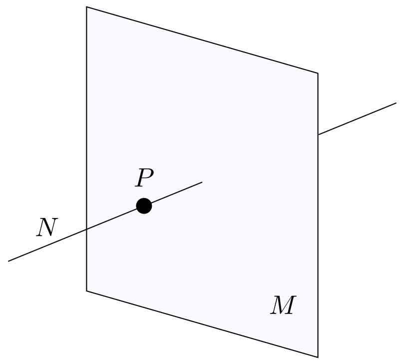

Figure 2. The algebraic variety $V(xy, xz)$

and the line

$$N = \{(x, y, z) : y = z = 0\}.$$

The fact that $\dim R = 2$ corresponds to the observation that the largest component (the plane) has degree 2.

The maximal ideal $I = (x - 1, y, z)$ corresponds to the point $P = (1, 0, 0) \in k^3$, which is on L. The dimension of the ideal I is $\dim R/I = 0$, since I is maximal. However, the height of I is only 1, since the only prime ideal contained in I is (y, z). Hence, the formula (3.1) breaks down. This is an illustration of the fact that height is a *local* notion: it only depends on the behavior of R around I. This can also be seen visually: while the dimension of R is 2, around the point $(1, 0, 0)$, R looks like a line, with dimension 1.

Example 3.5 (Nonreduced points)**.** Consider the ring

$$R = k[x]/(x^2).$$

The ideal (x^2) is not radical, and hence this is not the coordinate ring of any algebraic variety (since x^2 vanishes if and only if x does). This ring has only one prime ideal (x), which is maximal. Indeed, since $x \cdot x = 0$, any prime ideal must contain x. Hence, the Krull dimension of this ring is 0, since there cannot be any nontrivial chains of prime ideals. This may seem surprising, since the variable x "looks" like an additional variable. This illustrates that Krull dimension is insensitive to nilpotent elements. From the geometric perspective, $\mathrm{Spec}(R)$ is the single point (x), equipped with additional "infinitesimal" structure coming from the nilpotent element x. However, the dimension is still 0.

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