LAGRANGIAN AND HAMILTONIAN MECHANICS

SHIHAN KANUNGO

Abstract. In this expository paper, we discuss the basics of Lagrangian and Hamiltonian dynamics. We derive the Euler-Lagrange equations fro D'Alembert's principle, show that they are equivalent to Hamilton's principle of least actions, and finally use them to derive Hamilton's equations. We also provide some examples to illustrate the use of Lagrangian and Hamiltonian dynamics.

1. Holonomic constraints and degrees of freedom

Consider a system of N particles in three dimensional space, each with position vector $\boldsymbol{r}_i(t)$ for $i = 1, \ldots, N$. Note that each $\boldsymbol{r}_i(t) \in \mathbb{R}^3$ is a 3-vector. We thus need $3N$ coordinates to specify the system, this is the *configuration space*. *Newton's 2nd law* tells us that the equation of motion for the i-th particle is

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i^{\text{ext}} + \boldsymbol{F}_i^{\text{con}},$$

for $i = 1, \ldots, N$. Here $\boldsymbol{p}_i = m_i \boldsymbol{v}_i$ is the linear momentum of the i-th particle and $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ is its velocity. We decompose the total force on the i-th particle into an external force $\boldsymbol{F}_i^{\text{ext}}$ and a *constraint* force $\boldsymbol{F}_i^{\text{con}}$. By external forces we imagine forces due to gravitational attraction or an electro-magnetic field, and so forth.

By a constraint on a particles we imagine that the particle's motion is limited in some rigid way. For example the particle/bead may be constrained to move along a wire or its motion is constrained to a given surface. If the system of N particles constitute a rigid body, then the distances between all the particles are rigidly fixed and we have the constraint

$$|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| = c_{ij},$$

for some constants c_{ij}, for all $i, j = 1, \ldots, N$. All of these are examples of holonomic constraints.

Definition (Holonomic constraints). For a system of particles with positions given by $\boldsymbol{r}_i(t)$ for $i = 1, \ldots, N$, constraints that can be expressed in the form

$$g(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) = 0,$$

are said to be *holonomic*. Note they only involve the configuration coordinates.

We will only consider systems for which the constraints are holonomic. Systems with constraints that are non-holonomic are: gas molecules in a container (the constraint is only expressible as an inequality); or a sphere rolling on a rough surface without slipping (the constraint condition is one of matched velocities).

Let us suppose that for the N particles there are m holonomic constraints given by

$$g_k(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) = 0,$$

for $k = 1, \ldots, m$. The positions $\boldsymbol{r}_i(t)$ of all N particles are determined by $3N$ coordinates. However due to the constraints, the positions $\boldsymbol{r}_i(t)$ are not all independent. In principle, we

can use the m holonomic constraints to eliminate m of the $3N$ coordinates and we would be left with $3N - m$ independent coordinates, i.e. the dimension of the configuration space is actually $3N - m$.

Example (**Two particles connected by a light rod).** Suppose two particles can move freely in three-dimensional space, their position vectors at any time given by the vectors $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2 = \boldsymbol{r}_2(t)$, each with three components. Hence 6 pieces of information, the three components for each vector are required to specify the state of the system at any time t. The dimension of the configuration space is 6. Now suppose the two particles are connected by a light rigid rod of length ℓ. Thus for this system the vectors $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2 = \boldsymbol{r}_2(t)$ are restricted so that at any time t the constraint/condition

$$|\boldsymbol{r}_1(t) - \boldsymbol{r}_2(t)| = \ell$$

is satisfied. This constraint equation represents a single relation between the 6 configuration variables. In principle we can solve for any one of the configuration variables in terms of the other 5 configuration variables. Hence the system is constrained to evolve on a 5-dimensional submanifold of the 6-dimensional configuration space. The dimension of the configuration space is 5.

Definition (**Degrees of freedom).** The dimension of the configuration space is called the *number of degrees of freedom.*

Thus we can transform from the 'old' coordinates $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ to new *generalized coordinates* $q_1, \ldots, q_n$ where $n = 3N - m$:

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{r}_1(q_1, \ldots, q_n, t) \\ &\vdots \\ \boldsymbol{r}_N &= \boldsymbol{r}_N(q_1, \ldots, q_n, t) \end{aligned}$$

2. D'Alembert's Principle

We will restrict ourselves to systems for which the net work of the constraint forces is zero, i.e. we suppose

$$\sum_{i=1}^{N} \boldsymbol{F}_i^{\text{con}} \cdot \mathrm{d}\boldsymbol{r}_i = 0,$$

for every small change $\mathrm{d}\boldsymbol{r}_i$ of the configuration of the system (for t fixed). Recall that the work done by a particle is given by the force acting on the particle times the distance travelled in the direction of the force. So here for the i-th particle, the constraint force applied is $\boldsymbol{F}_i^{\text{con}}$ and suppose it undergoes a small displacement given by the vector $\mathrm{d}\boldsymbol{r}_i$. Since the dot product of two vectors gives the projection of one vector in the direction of the other, the dot product $\boldsymbol{F}_i^{\text{con}} \cdot \mathrm{d}\boldsymbol{r}_i$ gives the work done by $\boldsymbol{F}_i^{\text{con}}$ in the direction of the displacement $\mathrm{d}\boldsymbol{r}_i$.
If we combine the assumption that the net work of the constraint forces is zero with Newton's 2nd law

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i^{\text{ext}} + \boldsymbol{F}_i^{\text{con}},$$

from the last section, we find

$$\begin{aligned} & \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \left(\boldsymbol{F}_i^{\mathrm{ext}} + \boldsymbol{F}_i^{\mathrm{con}} \right) \cdot \mathrm{d}\boldsymbol{r}_i \\ \Leftrightarrow \qquad & \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i + \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{con}} \cdot \mathrm{d}\boldsymbol{r}_i \\ \Leftrightarrow \qquad & \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i. \end{aligned}$$

In other words we have

$$\sum_{i=1}^{N} \left(\dot{\boldsymbol{p}}_i - \boldsymbol{F}_i^{\mathrm{ext}} \right) \cdot \mathrm{d}\boldsymbol{r}_i = 0,$$

for every small change $\mathrm{d}\boldsymbol{r}_i$. This represents *D'Alembert's principle*. Note in particular that no forces of constraint are present.

Remark 1. The assumption that the constraint force does no net work is quite general. It is true in particular for holonomic constraints. For example, for the case of a rigid body, the internal forces of constraint do no work as the distances $|\boldsymbol{r}_i - \boldsymbol{r}_j|$ between particles is fixed, then $\mathrm{d}(\boldsymbol{r}_i - \boldsymbol{r}_j)$ is perpendicular to $\boldsymbol{r}_i - \boldsymbol{r}_j$ and hence perpendicular to the force between them which is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$. Similarly for the case of the bead on a wire or particle constrained to move on a surface — the normal reaction forces are perpendicular to $\mathrm{d}\boldsymbol{r}_i$.

This leads to Lagrange's equations of motion. Consider the transformation to generalized coordinates

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, \ldots, q_n, t),$$

for $i = 1, \ldots, N$. If we consider a small increment in the displacements $\mathrm{d}\boldsymbol{r}_i$ then the corresponding increment in the work done by the external forces is

$$\sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i,j=1}^{N,n} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \, \mathrm{d}q_j = \sum_{j=1}^{n} Q_j \, \mathrm{d}q_j.$$

Here we have used the chain rule

$$\mathrm{d}\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \, \mathrm{d}q_j,$$

and we set for $j = 1, \ldots, n$,

$$Q_j = \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}.$$

We think of the Q_j as *generalized forces*. We now assume the work done by these forces depends on the initial and final configurations only and not on the path between them. In other words we assume there exists a potential function $V = V(q_1, \ldots, q_n)$ such that

$$Q_j = -\frac{\partial V}{\partial q_j}$$

for $j = 1, \ldots, n$. Such forces are said to be *conservative*. We define the total *kinetic energy* to be

$$T := \sum_{i=1}^{N} \tfrac{1}{2} m_i \left| \boldsymbol{v}_i \right|^2,$$

and the *Lagrange function* or *Lagrangian* to be

$$\mathcal{L} := T - V.$$

Theorem 2 (**Lagrange's equations**). *D'Alembert's principle, under the assumption the constraints are holonomic, is equivalent to the system of ordinary differential equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0,$$

for $j = 1, \ldots, n$. These are known as Lagrange's equations *of motion.*

Proof. The change in kinetic energy mediated through the momentum — the first term in D'Alembert's principle — due to the increment in the displacements $\mathrm{d}\boldsymbol{r}_i$ is given by

$$\sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} m_i \dot{\boldsymbol{v}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i,j=1}^{N,n} m_i \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \, \mathrm{d}q_j.$$

From the product rule we know that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) &\equiv \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + \boldsymbol{v}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) \\ &\equiv \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j}. \end{aligned}$$

Also, by differentiating the transformation to generalized coordinates we see

$$\boldsymbol{v}_i \equiv \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \, \dot{q}_j \qquad \text{and} \qquad \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} \equiv \frac{\partial \boldsymbol{r}_i}{\partial q_j}.$$

Using these last two identities we see that

$$\begin{aligned} \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i &= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} m_i \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \mathrm{d}q_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j} \right) \right) \mathrm{d}q_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} \right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j} \right) \right) \mathrm{d}q_j \\ &= \sum_{j=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{N} \tfrac{1}{2} m_i \left| \boldsymbol{v}_i \right|^2 \right) \right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N} \tfrac{1}{2} m_i \left| \boldsymbol{v}_i \right|^2 \right) \right) \mathrm{d}q_j. \end{aligned}$$

Hence we see that D'Alembert's principle is equivalent to

$$\sum_{j=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right) \mathrm{d}q_j = 0.$$

Since the q_j for $j = 1, \ldots, n$, where $n = 3N - m$, are all independent, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0,$$

for $j = 1, \ldots, n$. Using the definition for the generalized forces Q_j in terms of the potential function V gives the result. □

Remark 3 (**Configuration space**). As already noted, the n-dimensional subsurface of $3N$-dimensional space on which the solutions to Lagrange's equations lie is called the *configuration space*. It is parameterized by the n generalized coordinates $q_1, \ldots, q_n$.

Example (**Simple harmonic motion**). Consider a particle of mass m moving in a one dimensional Hookeian force field $-kx$, where k is a constant. The potential function $V = V(x)$ corresponding to this force field satisfies

$$\begin{aligned} & -\frac{\partial V}{\partial x} = -kx \\ \Leftrightarrow \qquad & V(x) - V(0) = \int_0^x k\xi \,\mathrm{d}\xi \\ \Leftrightarrow \qquad & V(x) = \tfrac{1}{2}x^2. \end{aligned}$$

The Lagrangian $\mathcal{L} = T - V$ is thus given by

$$\mathcal{L}(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2.$$

From Hamilton's principle the equations of motion are given by Lagrange's equations. Here, taking the generalized coordinate to be $q = x$, the single Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0.$$

Using the form for the Lagrangian above we find that

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x},$$

and so Lagrange's equation of motion becomes

$$m\ddot{x} + kx = 0.$$

3. Hamilton's Principle

We consider mechanical systems with holonomic constraints and all other forces conservative. Recall, we define the *Lagrange function* or *Lagrangian* to be

$$\mathcal{L} = T - V,$$

where

$$T = \sum_{i=1}^{N} \tfrac{1}{2}m_i \left|\boldsymbol{v}_i\right|^2$$

is the total kinetic energy for the system, and V is its potential energy.

Definition (Action). If the Lagrangian $\mathcal{L}$ is the difference of the kinetic and potential energies for a system, i.e. $\mathcal{L} = T - V$, we define the *action* $S = S(\boldsymbol{q})$ from time t_1 to t_2, where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$, to be the functional

$$S(\boldsymbol{q}) := \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\mathrm{d}t.$$

Hamilton [1834] realized that Lagrange's equations of motion were equivalent to a variational principle.

Theorem 4 (Hamilton's principle of least action). *The correct path of motion of a mechanical system with holonomic constraints and conservative external forces, from time t_1 to t_2, is a stationary solution of the action. Indeed, the correct path of motion $\boldsymbol{q} = \boldsymbol{q}(t)$, with $\boldsymbol{q} = (q_1, \ldots, q_n)^T$, necessarily and sufficiently satisfies Lagrange's equations of motion for $j = 1, \ldots, n$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0.$$

It is Hamilton's form of the principle of least action "because in many cases the action of $\boldsymbol{q} = \boldsymbol{q}(t)$ is not only an extremal but also a minimum value of the action functional".

Proof. Suppose the boundary conditions are $\boldsymbol{q}(a) = A$ and $\boldsymbol{q}(b) = B$. the principle of least action states that the action S is extremized at q, so the functional

$$\int_a^b \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t$$

is extremized. In our proof, we assume that $\mathcal{L}$ is twice continuously differentiable, although the result is still true with a weaker assumption. Let $\widetilde{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon\boldsymbol{\eta}$ be a perturbation of $\boldsymbol{q}$, where ε is a small positive number and $\boldsymbol{\eta}$ is differentiable and $\boldsymbol{\eta}(a) = \boldsymbol{\eta}(b) = 0$. Then we have

$$\mathcal{L}\left(\widetilde{q}, \frac{\mathrm{d}\widetilde{\boldsymbol{q}}}{\mathrm{d}t}, t\right) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \varepsilon\left(\boldsymbol{\eta}_1\frac{\partial \mathcal{L}}{\partial q_1} + \cdots + \boldsymbol{\eta}_n\frac{\partial \mathcal{L}}{\partial q_n} + \dot{\boldsymbol{\eta}}_1\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \cdots + \dot{\boldsymbol{\eta}}_n\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right) + \mathcal{O}(\varepsilon^2),$$

where $\boldsymbol{\eta}_i(t)$ denotes the i'th coordinate of $\boldsymbol{\eta}(t)$. Then, we have

$$S(\widetilde{\boldsymbol{q}}) - S(\boldsymbol{q}) = \varepsilon\left[\int_a^b \boldsymbol{\eta}_1\frac{\partial \mathcal{L}}{\partial q_1} + \cdots + \boldsymbol{\eta}_n\frac{\partial \mathcal{L}}{\partial q_n} + \dot{\boldsymbol{\eta}}_1\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \cdots + \dot{\boldsymbol{\eta}}_n\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\,dt\right] + \mathcal{O}(\varepsilon^2).$$

The first term is the first variant of the action, denoted by δS. Then the the principle of least action states that $\delta S = 0$, so

$$\int_a^b \boldsymbol{\eta}_1\frac{\partial \mathcal{L}}{\partial q_1} + \cdots + \boldsymbol{\eta}_n\frac{\partial \mathcal{L}}{\partial q_n} + \dot{\boldsymbol{\eta}}_1\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \cdots + \dot{\boldsymbol{\eta}}_n\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\,dt = 0.$$

Then the components $\boldsymbol{\eta}_i\dfrac{\partial \mathcal{L}}{\partial q_i} + \dot{\boldsymbol{\eta}}_i\dfrac{\partial \mathcal{L}}{\partial \dot{q}_n}$ are independent are independent for different i, so we have

$$\int_a^b \boldsymbol{\eta}_i\frac{\partial \mathcal{L}}{\partial q_i} + \dot{\boldsymbol{\eta}}_i\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt = 0.$$

Then, let $u = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$, $v = \boldsymbol{\eta}_i$. Then

$$du = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt \quad \text{and} \quad dv = \dot{\boldsymbol{\eta}}_i\,dt.$$

Then integration by parts gives

$$\int_a^b \dot{\boldsymbol{\eta}}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt = \left[\boldsymbol{\eta}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right]_a^b - \int_a^b \boldsymbol{\eta}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt.$$

Now since $\boldsymbol{\eta}(a) = \boldsymbol{\eta}(b) = 0$, so the first term is zero. Then,

$$\int_a^b \dot{\boldsymbol{\eta}}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt = -\int_a^b \boldsymbol{\eta}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt.$$

Putting this into our original expression gives

$$\int_a^b \boldsymbol{\eta}_i \frac{\partial \mathcal{L}}{\partial q_i} - \boldsymbol{\eta}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,dt = 0.$$

For this to be true for all $\boldsymbol{\eta}$, we must have

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

for all $i = 1, \ldots, n$, which are the Euler-Lagrange equations. Reversing the steps above shows that the Euler-Lagrange equations imply that the action S is extrimized on the correct path. □

We define the *generalized momentum* $\boldsymbol{p} = (p_1, \ldots, p_n)$ by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

We define the *Hamiltonian* by

$$\mathcal{H} = \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - \mathcal{L}.$$

The Euler Lagrange equations can be written as

$$\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial q_i}$$

and

$$\boldsymbol{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

Then, we can write these equations in terms of the Hamiltonian as follows:

$$\frac{\mathrm{d}\boldsymbol{p}_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}$$

and

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}.$$

These are known as *Hamilton's Equations*.

Example (Simple harmonic oscillator). The Lagrangian for the simple harmonic oscillator, which consists of a mass m moving in a quadratic potential field with characteristic coefficient k, is

$$\mathcal{L}(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2.$$

The corresponding generalized momentum is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

which is the usual momentum. This implies $\dot{x} = p/m$ and so the Hamiltonian is given by

$$\begin{aligned}
\mathcal{H}(x, p) &= \dot{x}p - \mathcal{L}(x, \dot{x}) \\
&= \frac{p}{m}\, p - \left(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2\right) \\
&= \frac{p^2}{m} - \left(\tfrac{1}{2}m\left(\frac{p}{m}\right)^2 - \tfrac{1}{2}kx^2\right) \\
&= \frac{1}{2}\,\frac{p^2}{m} + \frac{1}{2}kx^2.
\end{aligned}$$

Note this last expression is the sum of the kinetic and potential energies and so $\mathcal{H}$ is the total energy. Hamilton's equations of motion are thus given by

$$\begin{aligned}
&\dot{x} = \partial\mathcal{H}/\partial p, \\
&\dot{p} = -\partial\mathcal{H}/\partial x
\end{aligned}
\qquad \Leftrightarrow \qquad
\begin{aligned}
&\dot{x} = p/m, \\
&\dot{p} = -kx.
\end{aligned}$$

Combining these two equations, we get the usual equation for a harmonic oscillator:

$$m\ddot{x} = -kx.$$

References

[1] L. Landau, and E. Lifshitz. (1976) *Course of Theoretical Physics. Volume 1, Mechanics.* Butterworth-Heinemann, 3rd Edition.

[2] Shankar, R. (2014) *Fundamentals of Physics: Mechanics, Relativity, and Thermodynamics.* Yale University Press, New Haven.

[3] J. Wang, B. Ricardo. (2018) *Competitive Physics: Mechanics and Waves.* World Scientific.

[4] Malham, Simon A. J. (2016) *An introduction to Lagrangian and Hamiltonian mechanics.* Online

[5] Kline, D. (2021) *Hamiltonian in Classical Mechanics.* University of Rochester, 30 June 2021. Link

[6] Mitofski, A.M. (2022) *Derivation of the Euler-Lagrange Equation.* Trine University, 22 May 2022. Link

[7] Wikipedia contributors. *Hamiltonian mechanics.* Wikipedia, The Free Encyclopedia, 14 May. 2024. Link

[8] Wikipedia contributors. *Lagrangian mechanics.* Wikipedia, The Free Encyclopedia, 12 May. 2024. Link